Not every matrix can be diagonalized. We will get as close as possible using an invertible matrix.

Recall the enemy:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable.

Definition: (block diagonal form)

Let $A \in M_n(\mathbb{C})$. A is in **block diagonal form** if $\exists$ $k \in \mathbb{N}$ and $n_1, n_2, \dots, n_k \in \mathbb{N}$ with $\sum_{i=1}^{k} n_i = n$ and

$$\boxed{A = \begin{bmatrix} A_1 & & & \\ & A_2 & & O \\ & & \ddots & \\ O & & & A_k \end{bmatrix}}$$

with $A_i \in M_{n_i}(\mathbb{C})$.

Example 1: $A \in M_6(\mathbb{C})$,

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ i & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5-3i & 1 & 16 \\ 0 & 0 & 0 & 15 & -7 & 32i \\ 0 & 0 & 0 & 12 & -4 & 16 \end{bmatrix}$$

is in block diagonal form.

$n_1 = 2, \quad n_2 = 1, \quad n_3 = 3$

Definition: (Jordan block)

If A is in block-diagonal form and A_i is a block of A, A_i is called a **Jordan block** if $\exists\, \lambda \in \mathbb{C}$ with

$$\boxed{A_i = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix}}$$

Idea: A Jordan block has the same entry on the diagonal and a 1 to the right of each diagonal entry, zeros everywhere else.

Block of size 1 = scalar

Theorem: (Jordan canonical form)

Let $A \in M_n(\mathbb{C})$. Then A is similar to a block diagonal matrix, all of whose blocks are Jordan blocks.

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Let $A \in M_n(\mathbb{C})$. Then A is similar to a block diagonal matrix, all of whose blocks are Jordan blocks.

We will first reduce to a simpler case, which will be used to obtain the general result.

Recall: (nilpotent matrix)

$A \in M_n(\mathbb{C})$ is called nilpotent if $\exists\ k \in \mathbb{N},\ A^k = 0$.

The smallest such k is called the index of A.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent of index 2 ;

$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Proposition: (eigenvalues of nilpotent)

Let $A \in M_n(\mathbb{C})$, suppose A is nilpotent. Then zero is the only eigenvalue of A.

proof: Zero is an eigenvalue of A since if $A \neq 0$, $\exists\, x \in \mathbb{C}^n$, $Ax \neq 0$.

But if $k =$ index of A, $A^k x = 0$.

If $A^{k-1}x \neq 0$, then

$y = A^{k-1}x$ is an eigenvector

for the eigenvalue zero.

If not, $\exists$ m, $1 \leq m < k-1$

such that

$$A^m x \neq 0, \text{ but}$$

$$A^{m+1} x = 0.$$

$\Rightarrow$ $A^m x$ is an eigenvector

for the eigenvalue zero.

Now suppose

$$Ax = \lambda x.$$

Then

$$\begin{aligned} A^k x &= A^{k-1}(Ax) \\ &= A^{k-1}(\lambda x) \\ &= A^{k-2}(\lambda A x) \\ &= A^{k-2}(\lambda^2 x) \\ &\ \vdots \\ &= \lambda^k x \end{aligned}$$

But $k =$ index of $A \Rightarrow A^k = 0$

$$\Rightarrow A^k x = 0.$$

Then $0 = \lambda^k x$

$\Rightarrow$ either $\lambda = 0$

or $x = 0$ and hence

is not an eigenvector.

Therefore, the only eigenvalue

for A is $\lambda = 0$.

$\square$

Proposition: (iterates) Suppose

$A \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$.

If $\exists\ k \in \mathbb{N}$, $k \geq 2$,

such that

$$A^k x = 0 \text{ but } A^{k-1} x \neq 0,$$

then $\{A^i x\}_{i=0}^{k-1}$ $(A^0 = I_n)$

is linearly independent.

<u>Proof</u>: Suppose $\exists$ such k, x for a given $A \in M_n(\mathbb{C})$.

Suppose $\exists \{\alpha_i\}_{i=0}^{k-1}$

with
$$\sum_{i=0}^{k-1} \alpha_i A^i x = 0.$$

Suppose $\exists\ j,\ 0 \leq j \leq k-1,$ $\alpha_j \neq 0$, and suppose $\alpha_i = 0\ \forall\ i < j$.

If $j=0$: Apply A to

$$\sum_{i=0}^{k-1} \alpha_i A^i x = 0.$$

$$A\left(\sum_{i=0}^{k-1} \alpha_i A^i x\right) = A0 = 0$$

$$\sum_{i=0}^{k-1} \alpha_i A^{i+1} x = 0$$

$$= \sum_{i=1}^{k} \alpha_{i-1} A^i x = 0$$

Then

$$\alpha_0 A x = -\sum_{i=2}^{k} \alpha_{i-1} A^i x$$

$$A x = -\sum_{i=2}^{k} \frac{\alpha_{i-1}}{\alpha_0} A^i x$$

Apply A^{k-2} to both sides:

$$A^{k-1} x = -\sum_{i=2}^{k} \frac{\alpha_{i-1}}{\alpha_0} A^{i+k-2} x$$

$$= -\sum_{i=0}^{k-2} \frac{\alpha_i}{\alpha_0} A^{k+i} x$$

But $A^{k+i}x = A^i(A^k x)$

$= A^i(0)$

$= 0 \quad (\forall\, 0 \leq i \leq k-2)$

$\Rightarrow Ax = 0$, contradiction.

Therefore, $\alpha_0 = 0$!

The same proof, using j instead of zero, will yield

$\alpha_j = 0 \quad \forall\, 1 \leq j \leq k-1$

$\Rightarrow \{A^i x\}_{i=0}^{k-1}$ is linearly independent $\square$

Then by the previous prop.,

$$\{A^i x\}_{i=0}^{k-1} \text{ is linearly independent in } \mathbb{C}^n.$$

Recall that the cardinality of any linearly independent set in $\mathbb{C}^n$ is less than or equal to n, contradiction since $k = |\{A^i x\}_{i=0}^{k-1}| > n$.

Therefore, $k \leq n$. $\square$

Corollary: (index) Let $A \in M_n(\mathbb{C})$ be nilpotent. Then the index of A is less than or equal to n.

proof: Suppose $k =$ index of A, $k > n$. Then $\exists\, x \in \mathbb{C}^n$,

$$A^k x = 0, \quad A^{k-1} x \neq 0.$$

(if not, then $A^{k-1} x = 0$ $\forall\, x \in \mathbb{C}^n \Rightarrow A^{k-1} = 0$, contradicts minimality of the index).

Jordan Canonical Form, Nilpotents

If $A \in M_n(\mathbb{C})$ is nilpotent, then A is similar to a block diagonal matrix, all of whose blocks are of the form

$$\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

($[0]$ in the case of a 1×1 block)